THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

SUNDAY, 14 DECEMBER, 1997

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Philosophy, Mathematics & Computation

Write your name and college of preference in BLOCK CAPITALS.

Name:

College of preference:

Attempt all the questions. Each of the ten parts of question 1 is worth 4 marks, and questions 2,3,4,5 are worth 15 marks each, giving a total of 100.

Answer question 1 on the grid overleaf. Write your answers to questions 2,3,4,5 in the space provided, continuing on the blank pages at the back of this booklet if necessary.

The use of calculators or formula sheets is NOT allowed.

Question 1

For each part on pages 4 and 5, exactly one of the answers (i), (ii), (iii), (iv) is correct. Indicate the correct answer with a tick (√) on the grid below.

Part	Answer			
	(i)	(ii)	(iii)	(iv)
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				
(g)				
(h)				
(j)				
(k)				

(a) The straight line in the (x, y) plane through the points $(-1, 3)$ and $(2, 1)$ is defined by the equation

(i) $3x + 2y = 3$, (ii) $-x + 3y = 1$, (iii) $2x + 3y = 7$, (iv) $x + 2y = 5$.

(b) There is a solution to the equation $x^3 + x^2 + 3 = 0$ between

(i) -2 and -1, (ii) -1 and 0, (iii) 0 and 1, (iv) 1 and 2.

(c) Anne, Bert, Clare, Derek and Emily are planning to play a game for which they need to divide themselves into three teams. Each team must have at least one member. The number of different ways they can do this is

(i) 10, (ii) 15, (iii) 25, (iv) 30.

(d) For the following statements

$$P: \frac{x(x-2)}{1-x} > 0, \qquad Q: 1 < x < 2$$

about a real number x,
(i) P implies Q, but Q does not imply P,
(ii) Q implies P, but P does not imply Q,
(iii) P implies Q, and Q implies P,
(iv) P and Q contradict each other.

(e) The least and greatest values of $\cos(\cos x)$ in the range $0 \le x \le \pi$ are

(i) 0 and 1, (ii) $-\cos 1$ and 1, (iii) -1 and 1, (iv) $\cos 1$ and 1.

(f) As the integer n becomes very large and positive,

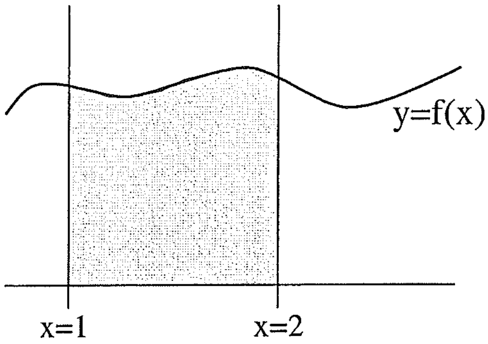
$$\frac{\sqrt{n} + (-1)^n}{\sqrt{n}}$$

(i) approaches (that is, converges to) 0,
(ii) approaches (that is, converges to) 1,
(iii) approaches infinity,
(iv) oscillates, but does not converge.

(g) The power of x which has the greatest coefficient in the expansion of $\left(1+\frac{1}{2}x\right)^{10}$ is

(i) x^2, (ii) x^3, (iii) x^5, (iv) x^{10}.

(h) The (shaded) area under the graph of $y = f(x)$ between $x = 1$ and $x = 2$ is given to be 1.



The area under the graph of $y = 2f(3-x)$ between $x = 1$ and $x = 2$ is therefore

(i) 1, (ii) 2, (iii) 3, (iv) 6.

(j) In a plane there are given n straight lines, no two of them parallel and no three of them meeting at a point. The number of parts they divide the plane into is

(i) $n+1$, (ii) n^2-n+2, (iii) $\frac{1}{2}n(n+1)+1$, (iv) 2^n.

(k) The simultaneous equations

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x + 3y - z &= 4 \\ 4x - y + 5z &= 6 \end{aligned}$$

have
(i) no solutions,
(ii) exactly one solution,
(iii) exactly three solutions,
(iv) infinitely many solutions.

Question 2

(a) Show that $x^2 + 4x + 4 \geq 0$ for all values of x.

(b) For which values of the constant a is there at least one solution, x, of the inequality

$$ax^2 + 4x + 3 \leq x^2 - 1?$$

(c) Suppose that $1 < a \leq 2$. Find all values of x for which the inequality in (b) holds.

Question 3

A square made of cardboard has sides of unit length and corners marked A, B, C, D.

(a) Show that there are eight different ways of placing the cardboard square so that it completely covers the square region in the (x, y) plane with corners at the points $(0,0), (0,1), (1,1), (1,0)$.

(b) Initially the square is positioned so that A, B, C, D are over the points $(0,0), (0,1), (1,1), (1,0)$, respectively. You may move the square by either

(i) rotating it in the plane by $90°$ about one of the corners, or

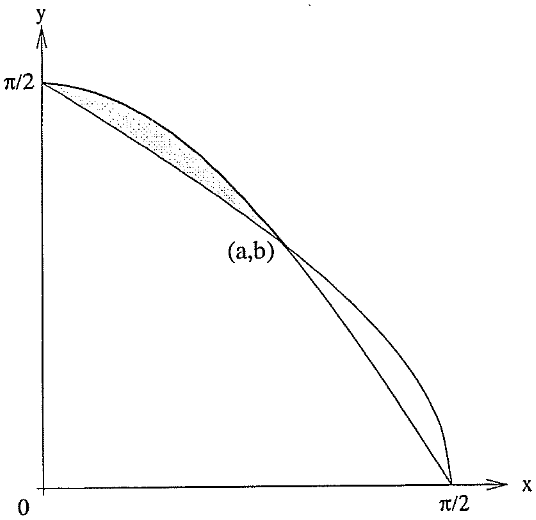
(ii) turning it over keeping one of the edges fixed in contact with the plane.

Show that after two moves it is possible to return the square to its initial position but with the corners B and D interchanged.

(c) Show that four of the eight configurations in (a) can be achieved from the initial position and moves described in (b).

Question 4

The curves $y = \frac{1}{2}\pi \cos x$ and $x = \frac{1}{2}\pi \cos y$ intersect at the three points $(0, \frac{1}{2}\pi)$, (a, b), $(\frac{1}{2}\pi, 0)$, as shown in the figure below.



(a) Explain why $a = b = \frac{1}{2}\pi \cos b$.

(b) Show that $\pi \sin b = \sqrt{\pi^2 - 4b^2}$.

(c) Show that the area of the shaded region is

$$\sqrt{\pi^2 - 4b^2} - \frac{\pi}{2} - b^2.$$

Question 5

Songs of the Martian classical period had just two notes (let us call them x and y) and were constructed according to rigorous rules:

I. the sequence consisting of no notes was deemed to be a song (perhaps the most pleasant);

II. a sequence starting with x, followed by two repetitions of an existing song and ending with y was also a song;

III. the sequence of notes obtained by interchanging x's and y's in a song was also a song.

All songs were constructed using those rules.

(a) Write down four songs of length 6 (that is, songs with exactly 6 notes).

(b) Show that if there are k songs of length m then there are $2k$ songs of length $2m+2$. Deduce that for each natural number n there are 2^n songs of length $2^{n+1}-2$.

Songs of the Martian later period were constructed using also the rule:

IV. if a song ended in y then the sequence of notes obtained by omitting that y was also a song.

(c) What lengths do songs of the later period have? That is, for which natural numbers n is there a song with exactly n notes? Justify your answer.